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Hydraulic jumps in an incompressible stratified fluid

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A multi-layer model is used to describe a 'two-dimensional' continuously stratified fluid. We use a momentum theorem in each layer to derive an ordinary differential equation describing the vertical structure behind a jump. This equation is compared with the corresponding equation for continuous flow. As one would expect from the classical one-layer theory, they are identical up to second order for weak disturbances. The energy change across a jump is also derived. By requiring that energy be lost through a jump, we calculate when a weak jump is possible in general. Algorithms for computing jumps of arbitrary strength are given. To ensure that the flow after the jump is stable and also for a numerical reason to be stated in § 8, we require that the Richardson number after the jump be equal to or greater than $\frac{1}{4}$. Numerical examples are given to show the range of parameters within which jumps are possible; the velocity profiles related to different kinds of jumps also appear. Since hydraulic jumps in a continuously stratified fluid have not yet been observed in any laboratory, it should be of interest to verify these calculations experimentally.

1. Introduction

We approximate two-dimensional flows with continuous density stratification and velocity shear by n layers of incompressible inviscid fluid. Each is characterized by a velocity, a thickness and a density. The first two quantities vary along streamlines while the last remains constant because of the assumption of incompressibility. We shall also assume hydrostatic balance. Therefore, when $n = 1$, we have the classical theory of a hydraulic jump in a single layer of homogeneous fluid. This theory was extended to two-layer fluid systems by Yih & Guha (1955), Houghton & Isaacson (1970), Long (1972) and Mehrotra & Kelly (1973). Their results are applicable to two-fluid systems with a discontinuous stratification and velocity profile. The idea of approximating a continuous stratification and shear by a sufficiently large number of layers of homogeneous fluid was first advanced by Benton (1953) for the calculation of the speed of linear long gravity waves. He proved for a certain class of base flows that the speeds obtained by his multi-layer formulation in the limit of zero thickness of each layer reduced to the solution of the differential equation derived for continuous stratification and shear by Long (1953). To justify a multi-layer descrip-

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tion of nonlinear flows, we demonstrate in the following section that in the case of jump-free flows the multi-layer equations reduce to the differential equation of Long in the limit of zero thickness. The numerical solution in a forthcoming paper by Lee & Su (1976) has also given us some confidence in the multi-layer description. In case of hydraulic jumps in a continuous stratification and shear, we start with an n-layer formulation, basically an extension of the two-layer model used by Yih & Guha (1955) and Houghton & Isaacson (1970). We work out the analytical properties of a weak jump, which turn out to be what one would expect from the classical theory of jumps. For example, the energy loss is of third order in the strength of the jump; infinitesimal jumps propagate with the speeds of linear gravity waves: in the neighbourhood of each of these speeds, finite but weak jumps are possible only if they propagate at supercritical speeds. We also derive in the limit of zero thickness a differential equation for the jump from the jump conditions in the n-layer formulation. This differential equation is compared with the corresponding differential equation for jump-free flows. In the limit of small disturbances, these two equations are identical up to second order. This is consistent with the finding that the energy loss in a weak jump is a third-order quantity.

Within the n-layer formulation, the governing equation becomes a system of n nonlinear algebraic equations. These may be viewed as the finite-element approximation. We devise a computational algorithm to solve these equations efficiently. A similar algorithm is used in Lee & Su to compute jump-free flows where n can be as large as 1000 with the computing time still relatively very short. In the jump problem, the computation is complicated by a branching process tied to the non-uniqueness of the conjugate state behind a jump. For $n = 2$ the non-uniqueness of the solution was first pointed out by Yih & Guha (1955). They circumvented the problem by requiring that the velocities after the jump in the two layers be identical. The problem was subsequently investigated by Long (1972), who suggested resolving it by experiment; Mehrotra & Kelly (1973) suggested taking only the solution which approaches an infinitesimally weak jump in the limit when the conditions upstream of the jump tend to the critical state. For large n, we found a large set of solutions. Most of these have extremely irregular velocity profiles even though we used a sufficient number of layers to represent the upstream velocity and density reasonably well. These irregular solutions are likely to be unstable owing to the very large shear, and the jump will become turbulent. Since the multi-layer formulation is questionable for describing turbulent mixing (among layers), we exclude all irregular solutions by requiring the Richardson number behind the jump to be greater than a certain value. Since the necessary condition for shear stability is not yet available, we take Miles' (1961) sufficient condition; i.e. the local Richardson number behind the jump is required to be greater than $\frac{1}{4}$. We have also tried different values of the Richardson number. As it decreases the number of possible jump solutions increases rapidly with n. In the data presented in the last section, when the upstream velocity is taken as uniform and the density as an exponentially decreasing function of height, we have used both $\frac{1}{4}$ and $\frac{1}{8}$ for the limiting Richardson number; the solution is found to be unique: the same in both cases. It is seen that the velocity profiles behind the jump in all the cases presented are very smooth. We have also found the range of a modified Froude number within which internal jumps (jumps which occur for a Froude number near those for internal gravity wave modes) are possible. However, external jumps are always possible under the free-surface condition provided that the Froude number ahead of the jump is greater than the critical Froude number of the surface wave. The interval wherein internal jumps are possible becomes smaller as one goes to higher modes. However for a given mode this range increases as the stratification increases. This is consistent with the expectation that it should shrink to zero as the stratification goes to zero.

The arrangement of the paper is clear from the title of each section. In the remainder of this section, we introduce the two basic equations in each layer. Under the assumptions made at the beginning of the section, and taking the vertical (z) axis to be the direction of the stratification and shear, we write the continuity equation and Bernoulli's theorem in each layer as

$$\partial h_i/\partial t + \partial(h_i u_i)/\partial x = 0, \tag{1}$$

$$\partial\phi_i/\partial t + \tfrac{1}{2}(\nabla\phi_i)^2 + gz + p/\rho_i = 0 \tag{2}$$

for $i = 1, 2, \ldots, n$. We have denoted the horizontal velocity by u_i, the velocity potential by ϕ_i and the thickness of the layer by $h_i(x,t)$. We suppose that the entire fluid is divided into n layers in the z direction, and shall denote the layer immediately above the ground by $i = 1$ and the top layer by $i = n$.

2. A digression on continuous flows

Before investigating discontinuous flows (jumps) in a stratified fluid, we make a short digression on continuous flows to demonstrate the utility of the multi-layer description of a stratified fluid. We suppose that the flow is steady. Hence in the ith layer we have for the equation of continuity

$$u_i = U_i H_i/h_i, \tag{3}$$

where we denote the values of u_i and h_i far upstream by U_i and H_i.

Applying the Bernoulli equation (2) in integrated form at the ith interface we obtain

$$\rho_{i+1}[\tfrac{1}{2}(u_{i+1}^2 + w_{i+1}^2) + gz_i] - \rho_i[\tfrac{1}{2}(u_i^2 + w_i^2) + gz_i] = \rho_{i+1}[\tfrac{1}{2}U_{i+1}^2 + gZ_i] - \rho_i[\tfrac{1}{2}U_i^2 + gZ_i],$$

where z_i and Z_i are the elevations of the ith interface at x and far upstream. Setting $\rho_{i+1} = \rho_i + \Delta\rho_i$, $u_{i+1} = u_i + \Delta u_i$ and $w_i = u_i\, dz_i/dx + O(h_i)$, we obtain

$$\Delta\left\{\frac{\rho_i}{2}u_i^2\left[1 + \left(\frac{dz_i}{dx}\right)^2\right] - \frac{\rho_i}{2}U_i^2\right\} + g\Delta\rho_i(z_i - Z_i) = 0. \tag{4}$$

Dividing the above equation by H_i, substituting for u_i from (3) and taking the limit $H_i = \Delta Z_i \to 0$, we obtain an equation due to Dubreil-Jacotin (1937) and Long (1953):

$$Z_{xx} + Z_{zz} + \tfrac{1}{2}(Z_x^2 + Z_z^2 - 1)\frac{d}{dZ}(\ln \rho U^2) + \frac{g}{\rho U^2}\frac{d\rho}{dZ}(z - Z) = 0. \tag{5}$$

For the purpose of comparison with the discontinuous flows later, we derive from (5) under the hydrostatic assumption, i.e. deletion of x derivatives, the following equation for $\zeta \equiv z - Z$ with the upstream elevation Z as the independent variable:

$$\frac{d}{dZ}\left\{\frac{\rho U^2}{2}\left[1-\left(1+\frac{d\zeta}{dZ}\right)^{-2}\right]\right\}-g\frac{d\rho}{dZ}\zeta = 0. \qquad (6)$$

This can also be obtained directly from (4) by dropping $(dz/dx)^2$, dividing through by ΔZ and taking the limit $\Delta Z \to 0$. The boundary conditions for steady flow over a barrier of profile $\zeta_0(x)$ are as follows:

$$\left.\begin{array}{r} \zeta = \zeta_0(x) \quad \text{at} \quad Z = 0, \\ \left.\begin{array}{r}\zeta = 0 \text{ for a rigid top surface} \\ \text{Pressure} = \text{constant for a free top surface}\end{array}\right\} \text{at } Z = H. \end{array}\right\} \qquad (7)$$

For flow problems in the real atmosphere $H \to \infty$. Neither a free nor a rigid surface as stipulated above is appropriate. In the linear approximation, it is known that the proper boundary condition at the top is a radiation condition (Eliassen & Palm 1961; Drazin & Su 1975). Since the corresponding condition for nonlinear problems has not yet been formulated, we shall restrict our subsequent investigation to the boundary conditions specified by (7).

Equation (6) with the boundary conditions (7) specifies a two-point boundary-value problem. However, by using the multi-layer description, it can be shown that the problem can be dealt with as a one-point initial-value problem. The numerical integration for continuous flows over a barrier is described in detail by Lee & Su (1976).

3. Discontinuous flows

We assume hydrostatic balance. Hence we neglect w_i compared with u_i in (2), which is then applied at the ith interface for fluid above and below to eliminate the pressure term. After some manipulation, we obtain

$$\frac{\partial}{\partial t}(\rho_i u_i)+\frac{\partial}{\partial x}\left[\rho_i\frac{u_i^2}{2}+g\rho_i(h_0+h_1+\ldots+h_i)+g\sum_{j=i+1}^{n}\rho_j h_j\right]+\frac{\partial p_s}{\partial x} = 0 \qquad (8)$$

for $i = 1, 2, \ldots, n$, where p_s is the surface pressure at the top of the fluid and h_0 denotes the ground elevation. In the subsequent analysis of jumps we shall consider a flat ground surface, i.e. $h_0 = 0$. We then derive from (8) in conjunction with (1) the following momentum equations:†

$$\frac{\partial}{\partial t}(\rho_i h_i u_i)+\frac{\partial}{\partial x}\left(\rho_i h_i u_i^2+\rho_i g\frac{h_i^2}{2}\right)+g\rho_i{\sum_{j=1}^{n}}'\rho_{ij}h_i\frac{\partial h_j}{\partial x}+h_i\frac{\partial p_s}{\partial x} = 0, \quad i = 1,\ldots,n, \qquad (9)$$

where we use

$$\rho_{ij} = \begin{cases}1, & j \leqslant i,\\ \rho_j/\rho_i, & j > i,\end{cases} \qquad (10)$$

† These equations can also be obtained directly from momentum conservation for a stream tube between x and $x+dx$.

and the prime on the summation sign indicates omission of the term with $j = i$. In discussing hydraulic jumps, we need to consider the total energy of the entire fluid. It can be shown from (1) and (8) that

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = -p_s \frac{\partial}{\partial t} \sum_{i=1}^{n} h_i, \tag{11}$$

where

$$E = \sum_{i=1}^{n} \rho_i h_i \left[\frac{u_i^2}{2} + g\frac{h_i}{2}\right] + g \sum_{i<j}^{n} \sum_{j=1}^{n} \rho_j h_i h_j, \tag{12a}$$

$$F = \sum_{i=1}^{n} \rho_i h_i u_i \left\{\frac{u_i^2}{2} + g \sum_{j=1}^{n} \rho_{ij} h_j + \frac{p_s}{\rho_i}\right\}. \tag{12b}$$

We can now write down the equations governing jumps and the total energy change across a jump. Denoting the flow variables upstream and downstream of a jump by upper- and lower-case letters, we obtain from the time-independent form of (1) and (9)

$$U_i H_i = u_i h_i, \tag{13}$$

$$h_i u_i^2 - H_i U_i^2 + g\frac{h_i + H_i}{2} \sum_{j=1}^{n} \rho_{ij}(h_j - H_j) + \frac{h_i + H_i}{2\rho_i}(p_s - P_s) = 0. \tag{14}$$

Since (9), the last two terms of which represent exchange of momentum between layers, is not in the form of a conservation law, we write down its last two terms across the jump by taking the mean value of h_i. Physically, this is equivalent to taking the mean hydraulic head over the jump section as suggested by Yih & Guha (1955). This was also done and briefly discussed by Houghton & Isaacson (1970).

Eliminating u_i from (13) and (14) and defining $\xi_i = h_i/H_i - 1$, we obtain the jump equations

$$F_i^2 \frac{2\xi_i}{(1+\xi_i)(2+\xi_i)} = \sum_{j=1}^{n} \rho_{ij} \frac{H_j}{H_i} \xi_j + \frac{1}{g\rho_i H_i}(p_s - P_s) \quad \text{for } i = 1, 2, \ldots, \tag{15}$$

where $F_i^2 = U_i^2/gH_i$ is the Froude number for the ith layer. From (15), it is easy to derive in the limit as $H_i \to 0$ the differential equation of a jump as

$$\frac{d}{dZ}\left\{\rho U^2 \frac{2d\zeta/dZ}{(1+d\zeta/dZ)(2+d\zeta/dZ)}\right\} - g\frac{d\rho}{dZ}\zeta = 0, \tag{16}$$

where ζ has the same meaning as in (6).

To second order in ζ, (16) is the same as (6), which was derived for continuous and energy-conserving flows. Therefore weak jumps are approximately energy conserving. In fact, we shall presently calculate the energy change over a jump and show that for a weak jump the energy loss is of third order in ζ as one would expect from the classical theory of jumps in a homogeneous fluid.

4. Energy loss over a jump

The term on the right-hand side of (11) vanishes both for a free upper boundary, for which $p_s = 0$, and for a rigid surface, for which

$$\frac{\partial}{\partial t} \sum_{i=1}^{n} h_i = 0.$$

Therefore it may be seen that the total energy change across a jump is given by the difference in the fluxes of energy F across the jump. Using (13) and (14) in (12b), we obtain

$$\Delta E = F_d - F_u = \frac{1}{2}\sum_{i=1}^{n} \rho_i H_i U_i^3 f_i(\xi_i), \tag{17}$$

where
$$f(\xi) = \frac{1}{(1+\xi)^2} + \frac{4\xi}{(1+\xi)(2+\xi)} - 1 = \frac{-\xi^3}{(1+\xi)^2(2+\xi)} \tag{18}$$

and the subscripts d and u indicate downstream and upstream values respectively. We list some obvious properties of $f(\xi)$ for $-1 < \xi < \infty$:†

(i) $f'(\xi) = -2\xi^2(3+2\xi)/(1+\xi)^3(2+\xi)^2 \leqslant 0$,

(ii) $f(\xi) \sim -\frac{1}{2}\xi^3$ as $\xi \to 0$,

(iii) $f(\xi) \gtrless 0$ for $\xi \lessgtr 0$.

Property (ii) verifies our statement about the energy loss in a weak jump. Also, it may be seen from property (iii) that $\Delta E > 0$ if $\xi_1 < 0$ for all i. In other words, a jump wherein every layer accelerates is energetically impossible.

The limit of the loss of energy through a jump as expressed by (17) for the case of a continuously stratified flow can also be found. Taking Z as the upstream vertical co-ordinate, we can write the sum in (17) as an integral as follows:

$$\Delta E = \frac{1}{2}\int_0^{Z_{\text{top}}} dZ \rho(Z) U^3(Z) f\left(\frac{d\zeta}{dZ}\right) \tag{19a}$$

$$\sim -\frac{1}{4}\int_0^{Z_{\text{top}}} dZ \rho(Z) U^3(Z) \left(\frac{d\zeta}{dZ}\right)^3 \quad \text{for weak jumps.} \tag{19b}$$

5. Waves of infinitesimal amplitude

To study waves of infinitesimal amplitude, one linearizes either (6) or (16) and obtains

$$\frac{d}{dZ}\left[\rho(U+C_n)^2 \frac{d\zeta_n}{dZ}\right] - g\frac{d\rho}{dZ}\zeta_n = 0, \tag{20}$$

$$\zeta_n = 0 \quad \text{at} \quad Z = 0,$$

$$\left.\begin{array}{ll} \zeta_n = 0 & \text{for a rigid surface} \\ (U+C_n)^2\, d\zeta_n/dZ - g\zeta_n = 0 & \text{for a free surface} \end{array}\right\} \quad \text{at } Z = H,$$

where we have replaced u by $U + C_n$. C_n denotes an eigenvalue for which the homogeneous system (20) has a non-trivial solution ζ_n. We shall assume that for all ρ and U studied there exists a real discrete set of C_n.

The linear problem can also be formulated in terms of discrete layers expressed by (15). By neglecting all but linear terms in (15), we obtain the following homogeneous linear equations for the ξ_i.

(i) For a free surface:

$$\sum_{j=1}^{n}\left(\rho_{ij}\frac{H_j}{H_i} - \delta_{ij}F_j^2\right)\xi_j = 0, \quad i = 1, \ldots, n, \tag{21}$$

† $\xi = -1$ corresponds to $h = 0$ and $u \to \infty$.

where we define $F_j^2 \equiv (U_j+C_n)^2/gH_j$. The eigenvalue C_n is obtained by requiring the determinant of the coefficient matrix in (21) to vanish. This has been done by Benton (1954).

(ii) For a rigid surface:

$$\left.\begin{aligned}\xi_i = \frac{\rho_{i+1}(U_{i+1}+C_n)^2}{\rho_i(U_i+C_n)^2}\xi_{i+1} - \frac{g}{(U_i+C_n)^2}\left(1-\frac{\rho_{i+1}}{\rho_i}\right)\sum_{j=i+1}^{n} H_j\xi_j, \quad i = 1,\ldots,n-1,\\ \sum_{j=1}^{n} H_j\xi_j = 0.\end{aligned}\right\} \qquad (22)$$

The values of C_n obtained above, except for the largest one in the free-surface case, which describes the surface wave, give the speeds of internal waves of infinitesimal amplitude.

6. Weak jumps

Assuming the existence of a discrete set of C_n as above, we can show that weak jumps are possible energetically for an upstream velocity $U+C$ provided that $C-C_n \geqslant 0$ in a small neighbourhood of each C_n. Taking the weak-jump solution as $\zeta = A\zeta_n$, it can be shown from the weakly nonlinear theory of Benjamin (1966; see also Drazin, Lee & Su 1974) that the strength of the jump, denoted by A, is related to $C-C_n$ by

$$A = (4J/3K)(C-C_n), \qquad (23)$$

where C denotes the speed of a jump moving into fluid with velocity U. $U+C$ then is the speed of fluid moving into a stationary jump. In (23),

$$K \equiv \int_0^{Z_{\text{top}}} \rho(U+C)^2\left(\frac{d\zeta_n}{dZ}\right)^3 dZ,$$

$$J \equiv \int_0^{Z_{\text{top}}} \rho(U+C)\left(\frac{d\zeta_n}{dZ}\right)^2 dZ.$$

Now the expression for the energy change through a jump in (19*b*) can be simplified if U = constant, i.e.

$$\Delta E = -\frac{U+c}{4}A^3K = -\frac{16(U+c)J^3}{27K^2}(C-C_n)^3. \qquad (24)$$

Thus in the neighbourhood of each C_n, a weak jump is energetically possible for supercritical flows and impossible for subcritical flows, provided that the velocity upstream of the jump is uniform. Below we shall locate for a certain density stratification the exact range of C wherein jumps are possible.

7. Algorithms for computing jumps

Free surface (or with a passive layer extending to $z \to \infty$)

We assume the fluid to be divided into n layers. On top of the nth layer, we suppose that there is a passive layer with constant density and velocity ρ_{n+1} and u_{n+1}. Without loss of generality, the latter is taken to be zero. A passive layer is frequently used in the meteorological literature and is a generalization of a free surface where $\rho_{n+1} = 0$. From the laws of hydrostatics, it is easy to show that

$$p_s - P_s = -g\rho_{n+1}(z_n - Z_n) = -g\rho_{n+1} \sum_{j=1}^{n} (h_j - H_j). \tag{25}$$

With this p_s, the right side of the energy equation (11) can be written as a time derivative and included in the term $\partial E/\partial t$. The energy loss through a jump given in (17) remains valid. Substituting the above expression in (15) and subtracting the two equations obtained from the latter with free indices i and $i+1$, we have

$$\alpha(\xi_i) = (1-\rho_{n+1}/\rho_i)(1-\rho_{n+1}/\rho_{i+1})^{-1} U_{i+1}^2 U_i^{-2} \alpha(\xi_{i+1}) - gU_i^{-2}\left(1-\frac{\rho_{i+1}}{\rho_i}\right) \sum_{j=i+1}^{n} (\rho_j - \rho_{n+1})(\rho_{i+1}-\rho_{n+1})^{-1} H_j \xi_j \tag{26}$$

for $i = 1, 2, \ldots, n-1$, where

$$\alpha(\xi) = 2\xi/(1+\xi)(2+\xi). \tag{27}$$

For $i = n$, we obtain from (15) and (25)

$$D_f \equiv \frac{U_n^2}{g}\left(1-\frac{\rho_{n+1}}{\rho_n}\right)^{-1} \alpha(\xi_n) - \sum_{j=1}^{n} H_j \xi_j = 0. \tag{28}$$

The algorithm for computing all the ξ_i satisfying (26)–(28) is as follows. First pick a ξ_n in $-1 < \xi_n < \infty$ ($\xi_n \leqslant 1$ is excluded since it gives negative velocities), then compute $\xi_{n-1}, \xi_{n-2}, \ldots, \xi_1$ successively by letting i in (26) vary through the sequence $n-1, n-2, \ldots, 1$. In this way D_f in (28) can be taken as a function of ξ_n only. The zeros of D_f with the corresponding set of ξ_i's represent the solutions for the jumps.

Rigid surface

A similar algorithm is obtained for a rigid top. Here we have as an upper boundary condition

$$D_r \equiv \sum_{j=1}^{n} (h_j - H_j) = \sum_{j=1}^{n} H_j \xi_j = 0. \tag{29}$$

Eliminating the pressures in (15) we obtain

$$\alpha(\xi_i) = \frac{\rho_{i+1} U_{i+1}^2}{\rho_i U_{i+1}^2} \alpha(\xi_{i+1}) - \frac{g}{U_i^2}\left(1-\frac{\rho_{i+1}}{\rho_i}\right) \sum_{j=i+1}^{n} H_j \xi_j \tag{30}$$

for $i = 1, 2, \ldots, n-1$. To compute the jump, we proceed exactly as for a free surface, but use (30) instead of (26), and (29) instead of (28).

8. Limit of zero stratification

For zero density stratification and $U_i =$ constant, both algorithms presented above, i.e. (26) and (30), are extremely simple and reduce to $\alpha(\xi_i) = \alpha(\xi_{i+1})$ or $\alpha(\xi_1) = \alpha(\xi_2) = \ldots = \alpha(\xi_n)$. Depending on the value of ξ_n we have the following two cases.

(i) $-1 < \xi_n < 0$: no jump. It is easy to see from (27) that in this case $-\infty < \alpha(\xi_n) < 0$. Since ξ has a single value for each α in this range, we have

$\xi_i = \xi_n$ for $i = 1, 2, \ldots, n-1$. Since all the ξ_i are negative, the energy increases through the jump, which is impossible.

(ii) $\xi_n \geqslant 0$, i.e. $0 \leqslant \alpha_n \leqslant 2(1-2^{\frac{1}{2}})^2$. For any ξ_n in this range, there are two possible ξ_i's:

$$\xi_i = \begin{cases} \xi_n = \alpha_n^{-1} - \frac{3}{2} + [(\alpha_n^{-1} - \frac{3}{2})^2 - 2]^{\frac{1}{2}}, \\ \xi_n^* = \alpha_n^{-1} - \frac{3}{2} - [(\alpha_n^{-1} - \frac{3}{2})^2 - 2]^{\frac{1}{2}}, \end{cases} \tag{31}$$

where for definiteness we use the positive sign in front of the square-root sign to define ξ_n, which is the 'rational conjugate' of ξ_n^* as given in (31). Since all the ξ_i are positive, this is energetically favourable. For a rigid surface, we see that these positive ξ_i's fail to satisfy (29). Thus there is no jump, as one would expect for a rigid surface. However, it is not difficult to show that in the case of a free surface D_f in (28) can be made to vanish in many different ways. All the solutions for a given value of U^2/gH are discontinuous except the one which has a constant velocity profile. It can be shown that this continuous solution has the greatest energy loss across the jump among all the possible solutions for a given Froude number U^2/gH.

This non-uniqueness, stemming from the two possible solutions for ξ for a given value of α, is a characteristic of (26) and (30) for general upstream density and velocity profiles. As it is, the algorithm in the previous section may produce a highly discontinuous downstream velocity profile and it could become worse the thinner one makes the layer. To render the problem unique, or to avoid discontinuities in the velocity profile behind a jump, it is proposed in the present calculation that the Richardson number, defined as

$$Ri = -\frac{g}{\rho}\frac{d\rho}{dz}\bigg/\left(\frac{du}{dz}\right)^2,$$

be greater than a certain positive number. Guided by stability considerations (Miles 1961), we shall take $Ri > \frac{1}{4}$. This eliminates a very large number of branches in the algorithm of (26) and (30) and thus greatly facilitates the computation. Of course, we pay for this convenience by losing possible hydraulic jumps other than those calculated in the next section. However, those jumps for which the Richardson number is less than $\frac{1}{4}$ are likely to be unstable and thus have to be treated with proper turbulent mixing. From the mathematical point of view the introduction of the Richardson number here is a way of obtaining a smooth solution. We have used only two lower bounds on Ri ($\frac{1}{4}$ and $\frac{1}{8}$) for all the numerical examples. For such close bounds on Ri the results are identical, i.e. there is no solution for $Ri > \frac{1}{8}$ apart from that for $Ri > \frac{1}{4}$.

9. Numerical examples

We consider exponential density stratification and uniform velocity upstream of the jump. In this case the linear problem (20) can be solved explicitly. We obtain the eigenvalues governed by

$$lH \tan(lH) = \beta H \quad \text{(free surface)}$$

or

$$lH = n\pi, \quad n = 1, 2, \ldots \quad \text{(rigid surface)},$$

βH	Upper boundary condition	Mode number 0	1	2	3
0·1686	Free surface	$lH_L = 0$	$lH_L = 3·1890$	$lH_L = 6·3088$	$lH_L = 9·4425$
		$lH_U = 0·3994$	$lH_U = 3·1943$	$lH_U = 6·3099$	$lH_U = 9·4426$
	Rigid top		$lH_L = 3·1410$	$lH_L = 6·28318$	$lH_L = 9·4246$
			$lH_U = \pi$	$lH_U = 2\pi$	$lH_U = 3\pi$
1·686	Free surface	$lH_L = 0$	$lH_L = 3.2916$	$lH_L = 6·4024$	$lH_L = 9·5506$
		$lH_U = 1·0247$	$lH_U = 3·5815$	$lH_U = 6·5356$	$lH_U = 9·5987$
	Rigid top		$lH_L = 3·0962$	$lH_L = 6·2821$	$lH_L = 9·4229$
			$lH_U = \pi$	$lH_U = 2\pi$	$lH_U = 3\pi$

TABLE 1. Ranges of values of lH for jumps (possible for $lH_L \leqslant lH \leqslant lH_U$). Note that the values of lH_U are critical states in the sense of linear theory.

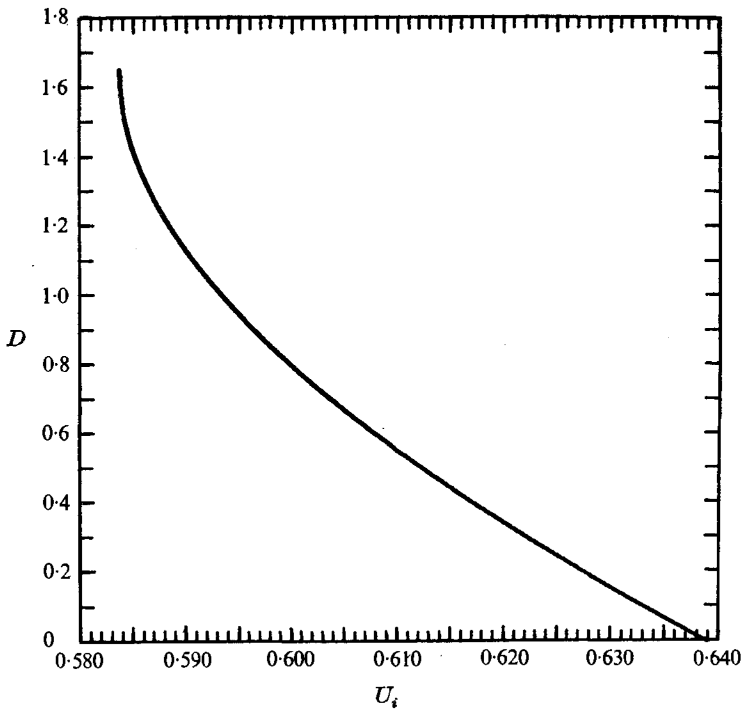


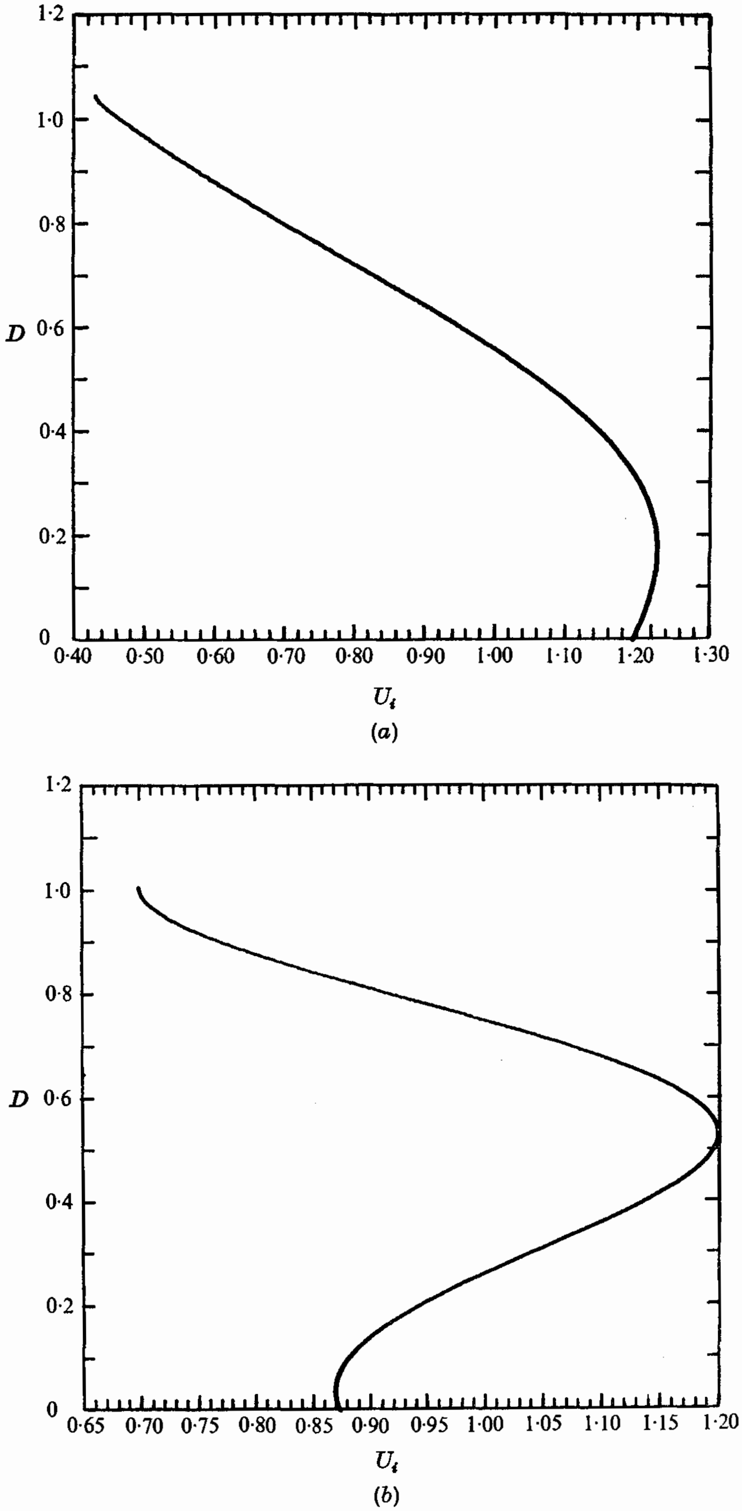
FIGURE 1. Velocity profile for $\beta H = 0·1686$, $lH = 0·2674$.

where

$$l^2 = \frac{g\beta}{U^2} - \frac{\beta^2}{4}, \quad \beta = -\frac{1}{\rho}\frac{d\rho}{dz}$$

and H is the total thickness of the fluid.

It can be shown with a proper non-dimensionalization that the nonlinear jump problem is characterized in the present case by the two non-dimensional parameters βH and lH. In our computation, we use the algorithms developed in § 7, take two values of βH and find the range of values of lH wherein a jump is possible energetically as in table 1. In this table, we denote surface waves by the



FIGURES 2 (*a*, *b*). For legend see next page.

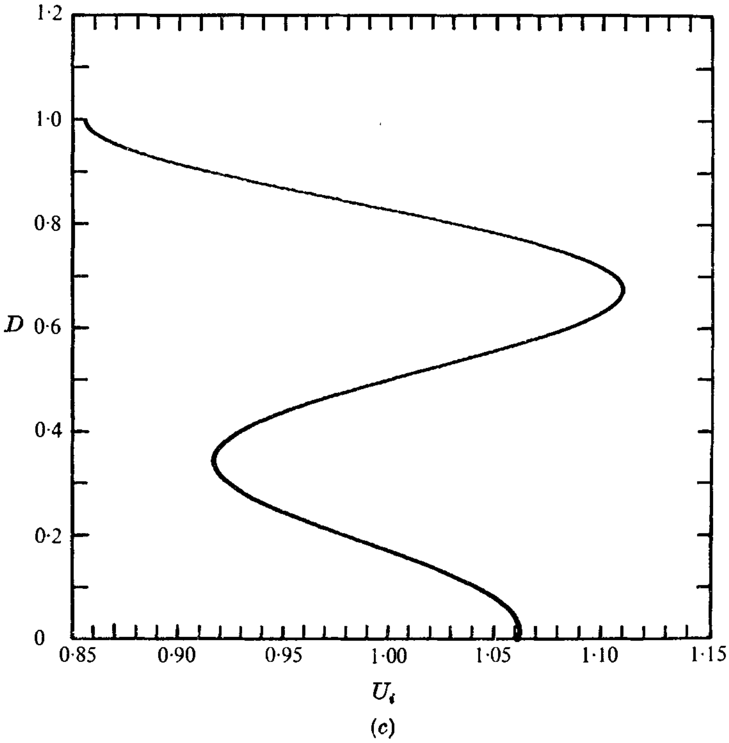


FIGURE 2. Velocity profiles for free surface. (a) $\beta H = 1{\cdot}686$, $lH = 3{\cdot}2916$; (b) $\beta H = 1{\cdot}686$, $lH = 6{\cdot}4024$; (c) $\beta H = 1{\cdot}686$, $lH = 9{\cdot}5506$.

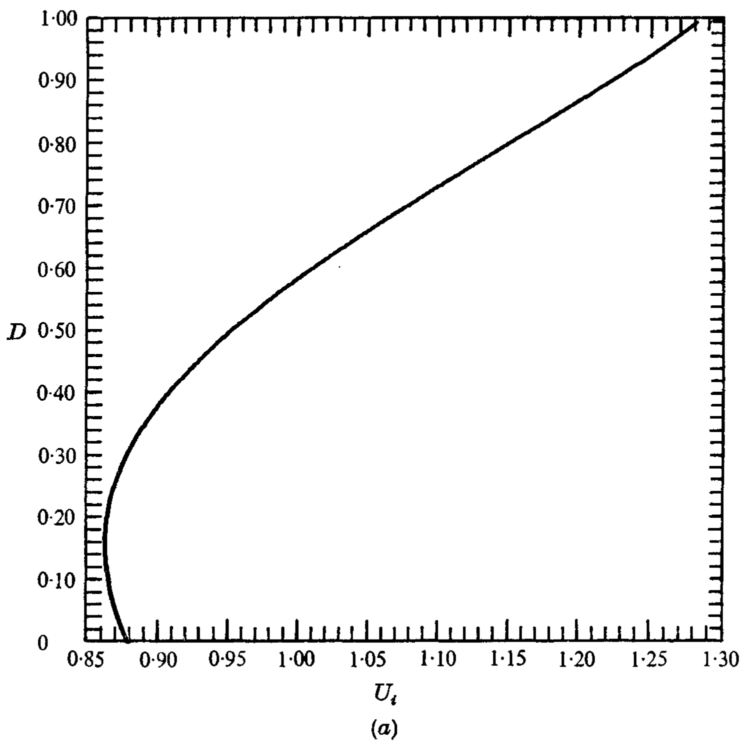
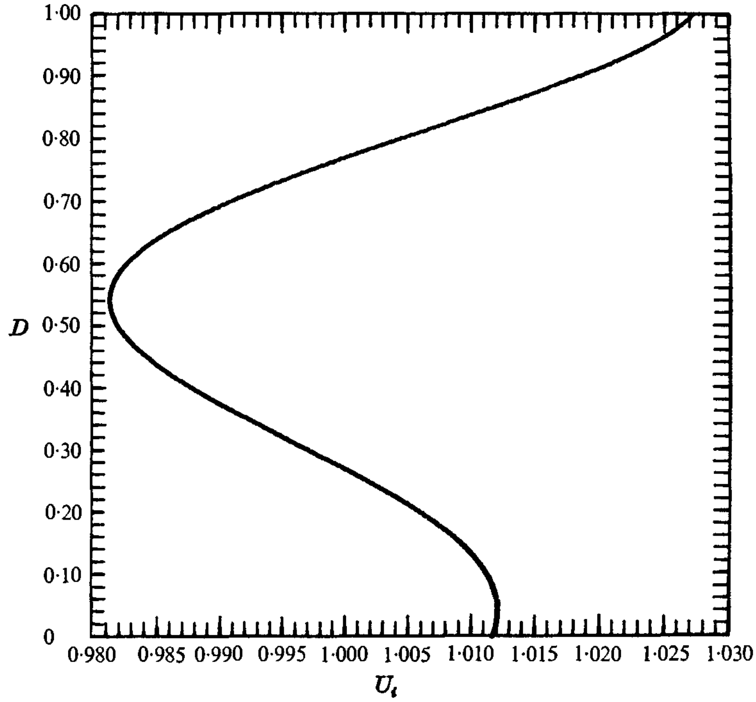
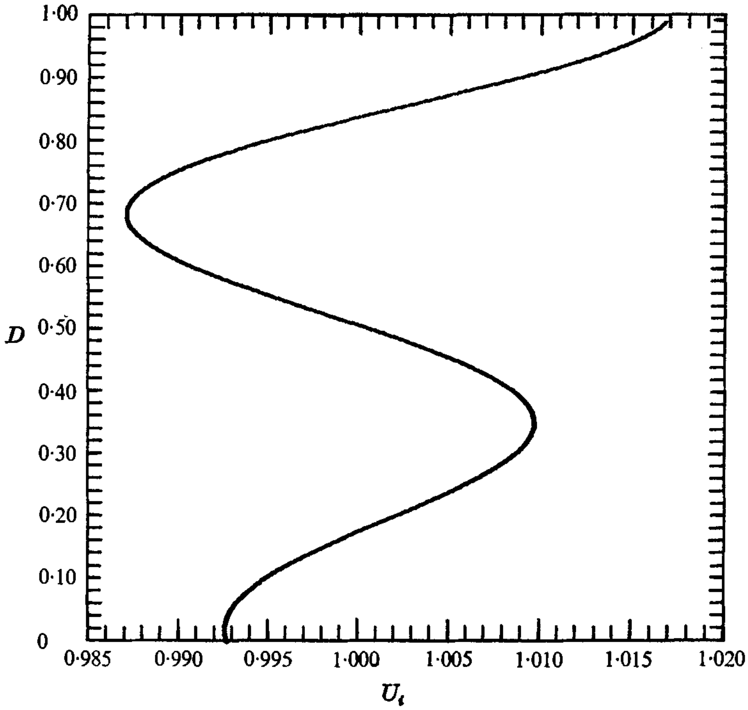


FIGURE 3(a). For legend see facing page.



(*b*)



(*c*)

FIGURE 3. Velocity profiles for rigid surface. (*a*) $\beta H = 1{\cdot}686$, $lH = 3{\cdot}0962$; (*b*) $\beta H = 1{\cdot}688$, $lH = 6{\cdot}2820$; (*c*) $\beta H = 1{\cdot}686$, $lH = 9{\cdot}44229$.

zeroth mode and internal waves by the first, second or third mode. We see that an external jump is possible if lH is less than a certain critical value. Put another way, the Froude number U^2/gH must be greater than a certain critical value to have an external jump as is the case for classical hydraulic jumps in homogeneous fluids. A typical velocity profile behind such a jump is presented in figure 1, where both the depth and the velocity are normalized by their upstream value. It may be seen that the particles near the surface decelerate more than those near the bottom. This is true for all the cases we have calculated for a free surface. For a rigid surface the opposite is true.

From table 1 it may be seen that the bands of values of lH wherein an internal jump is possible are quite narrow. This may explain why an internal hydraulic jump has not yet been observed experimentally in a continuously stratified fluid (Yih 1965, p. 130). In figure 2 we show the velocity profile after a jump for a free surface in the neighbourhood of the first and second modes. In figure 3 we show the velocity profile around the first, second and third modes in the case of a rigid surface. These diagrams were produced by a standard NCAR graphics routine. In most cases, the variation in velocity is greatly exaggerated. In the computation we found that the number of layers sufficient to approximate a continuous stratification increases with the value of lH. This seems to be a reasonable trend in view of more complicated structure of the velocity profile for higher values of lH. In our experiment we used 600–1000 layers. Even though the number of layers used was large, the computational time involved was still very short because of the simple algorithms as shown in § 4.

In conclusion we point out that jumps associated with the zeroth mode are always possible, while internal jumps occur only in narrow intervals on the lH scale. The width of these intervals increases as the stratification of the fluid increases. For the second βH we used above, it seems that these intervals are sufficiently wide to permit experimental verification.

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